## Proving the Termination of Narrowing by Proving the Termination of Rewriting

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(Joint work with Naoki Nishida, University of Nagoya, Japan)

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Termination of Narrowing

#### Outline



narrowing

#### termination of narrowing via termination of rewriting

- data generators
- main result

#### 3 automating the termination analysis

- abstract terms and argument filterings
- a direct approach to termination analysis
- a transformational approach

#### the technique in practice

- the termination tool TNT
- inference of safe argument filterings
- some refinements

#### related work

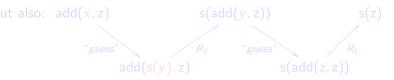
onclusions

Standard definition of addition (TRS)

$$\begin{array}{ccc} \mathsf{add}(\mathsf{z},y) & \to & y & (R_1) \\ \mathsf{add}(\mathsf{s}(x),y) & \to & \mathsf{s}(\mathsf{add}(x,y)) & (R_2) \end{array}$$

With rewriting:  $add(s(z), z) \rightarrow_{R_2} s(add(z, z)) \rightarrow_{R_1} s(z)$ 

With **narrowing**:  $add(s(z), z) \rightsquigarrow_{R_2} s(add(z, z)) \rightsquigarrow_{R_1} s(z)$ 

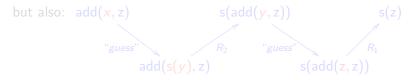


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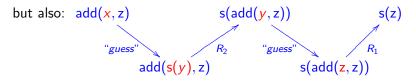


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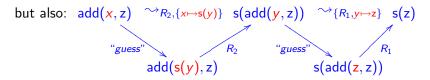


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#### Formal definition

#### Definition (rewriting)

$$(s \rightarrow_{p,R} s[r\sigma]_p)$$
 if there are

- $\begin{cases} \bullet \text{ a position } p \text{ of } s \\ \bullet \text{ a rule } R = (I \rightarrow r) \text{ in } \mathcal{R} \\ \bullet \text{ a substitution } \sigma \text{ such that } s|_p = I\sigma \end{cases}$

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## Definition (narrowing) $\underbrace{(s \leadsto_{p,R,\sigma} (s[r]_p)\sigma)}_{\text{if there are}} \text{ if there are } \begin{cases} \bullet \text{ a nonvariable position } p \text{ of } s \\ \bullet \text{ a variant } R = (l \to r) \text{ of a rule in } \mathcal{R} \\ \bullet \text{ a substitution } \sigma \text{ such that } s|_p \sigma = l\sigma \\ [\sigma = mgu(s|_p, l)] \end{cases}$

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#### Some motivation

We want to analyze the termination of narrowing

#### Why?

- narrowing is relevant in a number of areas: functional logic languages, partial evaluation, protocol verification, type inference, etc
- no termination prover for narrowing

We want to analyze the termination of narrowing by analyzing the termination of rewriting

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• many techniques and tools for rewriting!

#### Main ideas

- replace logic variables by data generators
- analyze the termination of rewriting with data generators
- adapt direct and transformational approaches, ,

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#### Termination of narrowing

#### The termination problem

• given a TRS, are all possible narrowing derivations finite?

Too strong!

 $\mathsf{add}(x,y) \sim_{R_2,\{x \mapsto \mathsf{s}(x')\}} \mathsf{add}(x',y) \sim_{R_2,\{x' \mapsto \mathsf{s}(x'')\}} \cdots$ 

#### In this work

given a TRS *R* and a set of terms *T*, are all possible narrowing derivations t<sub>1</sub> → t<sub>2</sub> → ... for t<sub>1</sub> ∈ *T* finite? (in symbols: *T* is →<sub>*R*</sub>-terminating)

For instance,  $\{ \mathsf{add}(s,t) \mid s \text{ is ground } \}$  is  $\rightsquigarrow_{\mathcal{R}}$ -terminating

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#### Theorem

*T* is  $\rightsquigarrow_{\mathcal{R}}$ -terminating if { $t\sigma \mid t \in T$  and  $t \rightsquigarrow_{\sigma}^* s$  in  $\mathcal{R}$ } is finite and  $\rightarrow_{\mathcal{R}}$ -terminating

#### **Drawbacks:**

- very difficult to approximate
- sufficient but not necessary:



The set  $\{f(x)\}$  is  $\rightsquigarrow_{\mathcal{R}}$ -terminating however  $\{f(a)\}$  is finite but not  $\rightarrow_{\mathcal{R}}$ -terminating:

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#### A first solution)

Variables in narrowing can be seen as generators of possibly infinite terms

Therefore 
$$\{t\sigma \mid t \in T \text{ and } t \sim_{\sigma}^{*} s \text{ in } \mathcal{R}\}$$

 $\{t\sigma \mid t \in T \text{ and } \sigma \text{ maps variables to possibly infinite terms } \}$ 

Why infinite terms?

#### Example

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•  $\operatorname{add}(x, z)$  is  $\to_{\mathcal{R}}$ -terminating for any  $\sigma$  mapping x to a finite term

 however, if σ maps x to an infinite term of the form s(s(...)), then the derivation for add(x, z)σ is now infinite:

$$\mathsf{add}(\mathsf{s}(\mathsf{s}(\ldots)),\mathsf{z}) o_{\mathcal{R}} \mathsf{s}(\mathsf{add}(\mathsf{s}(\mathsf{s}(\ldots)),\mathsf{z})) o_{\mathcal{R}} \ldots$$

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proving that the set

 $\{t\sigma \mid t \in T \text{ and } \sigma \text{ maps variables to possibly infinite terms } \}$ 

is  $\rightarrow_{\mathcal{R}}$ -terminating is often **too strong**...

#### **Example** Given the TRS

 $a \rightarrow a$  $f(x) \rightarrow x$ 

$$\begin{split} & \mathsf{f}(x) \text{ is clearly} \sim_{\mathcal{R}} \text{-terminating} \\ & \mathsf{but} \; \exists \sigma \; \mathsf{such} \; \mathsf{that} \; \mathsf{f}(x) \sigma \; \mathsf{is not} \; \rightarrow_{\mathcal{R}} \text{-terminating} \\ & (\mathsf{e.g.}, \; \sigma = \{ x \mapsto x \in \mathcal{R} \} \end{split}$$

 $\Rightarrow$  an infinite computation f(a)  $\rightarrow_{\mathcal{R}} f(a) \rightarrow_{\mathcal{R}} \dots$  is introduced by  $\sigma$  !!

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#### (A second solution)

#### $\Rightarrow$ forbid the reduction of redexes introduced by $\sigma...$

#### A second problem...

... this restriction makes the condition unsound!

**Example** Given the TRS

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- c(y, f(y))σ is →<sub>R</sub>-terminating if the reduction of the terms introduced by σ is forbidden
- but c(y, f(y)) is not  $\sim_{\mathcal{R}}$ -terminating!!

 $(e.g., c(y, f(y)) \sim_{\{y \mapsto a\}} c(a, c(b, b)) \sim_{id} c(a, c(b, b)) \sim_{id} \ldots)$ 

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#### Last (good) solution

 $\Rightarrow \left\{ \begin{array}{l} \mbox{restrict to narrowing derivations} \\ \mbox{where terms introduced by instantiation cannot be narrowed!} \end{array} \right.$ 

For instance,

- (innermost) basic narrowing over arbitrary TRSs
- lazy and needed narrowing over left-linear constructor TRSs

• . . .

### Any narrowing strategy over left-linear constructor TRSs can only introduce constructor substitutions $\Longrightarrow$

In the following, we consider left-linear constructor TRSs:

$$f_1(t_{11},\ldots,t_{1m_1}) \rightarrow r_1$$
  
 $\ldots$   
 $f_n(t_{n1},\ldots,t_{nm_n}) \rightarrow r_n$ 

with

- $f_i(t_{i1}, \ldots, t_{in_i})$  linear (no multiple occurrences of the same variable)
- $t_{i1}, \ldots, t_{in_i}$  constructor terms (no occurrence of  $f_1, \ldots, f_n$ )

 Property
 variables are bound to (irreducible) constructor terms

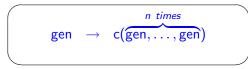
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 Our approach
 we replace variables by "data generators"

 that only produce (ground) constructor terms

#### Data generators [Antoy, Hanus, 2006; de Dios-Castro, López-Fraguas 2006]

For every TRS  $\mathcal{R},$  we define  $\mathcal{R}_{gen}$  as  $\mathcal{R}$  augmented with



for all constructor  $c/n \in \mathcal{C}, \ n \ge 0$ 

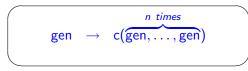
E.g., for  $\mathcal{C} = \{z/0, s/1\}$ , we have

$$\mathcal{R}_{gen} = \mathcal{R} \cup \left\{ \begin{array}{ll} gen & \rightarrow & z \\ gen & \rightarrow & s(gen) \end{array} \right\}$$

**Some notation:**  $\hat{t} = t\sigma$ , with  $\sigma = \{x \mapsto \text{gen} \mid x \in \mathcal{V}ar(t)\}$ 

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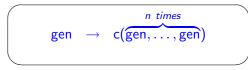
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#### Correctness of data generators

#### Completeness

$$\mathsf{If} \ s \leadsto_\sigma t \ \mathsf{in} \ \mathcal{R} \quad \mathsf{then} \quad \widehat{s} \to_{\mathsf{gen}}^* \widehat{s\sigma} \to \widehat{t} \ \mathsf{in} \ \mathcal{R}_{\mathsf{gen}}$$

#### Soundness is preserved for admissible derivations

• a derivation is admissible iff all the occurrences of gen originating from the replacement of the same variable are reduced to the same term

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#### Generally unsound

$$\mathsf{E.g., } \mathsf{add}(\mathsf{gen}, \mathsf{gen}) \to \mathsf{add}(\mathsf{z}, \mathsf{gen}) \to \mathsf{gen} \to \mathsf{s}(\mathsf{gen}) \to \mathsf{s}(\mathsf{z})$$

but  $\operatorname{add}(x, x) \sim_{\{x \mapsto z\}} z$  $\operatorname{add}(x, x) \sim_{\{x \mapsto s(x')\}} s(\operatorname{add}(x', s(x'))) \sim_{\{x' \mapsto z\}} s(s(z))$ ....

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#### What about termination in $\mathcal{R}_{gen}$ ?

#### Clearly, no term with occurrences of gen terminates!

Fortunately, relative termination of  $\mathcal{R}_{\text{gen}}$  suffices:

*T* is relatively *R*<sub>gen</sub>-terminating to *R* if every derivation *t*<sub>1</sub> → *t*<sub>2</sub>... for *t*<sub>1</sub> ∈ *T* contains finitely many →<sub>*R*</sub> steps

Theorem (termination of narrowing via termination of rewriting)

Let  $\mathcal{R}$  be a left-linear constructor TRS T is  $\sim_{\mathcal{R}}$ -terminating  $\widehat{T}$  is relatively to reminating to  $\mathcal{R}$ 

T is relatively  $\rightarrow_{\mathcal{R}_{gen}}$ -terminating to  $\mathcal{R}$ 

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Let  $\mathcal{R}$  be a left-linear constructor TRS T is  $\rightsquigarrow_{\mathcal{R}}$ -terminating iff  $\widehat{T}$  is relatively  $\rightarrow_{\mathcal{R}_{gen}}$ -terminating to  $\mathcal{R}$  w.r.t. admissible derivations

#### ⇒ sufficient condition

# What about termination in $\mathcal{R}_{gen}$ ?

Clearly, no term with occurrences of gen terminates!

Fortunately, relative termination of  $\mathcal{R}_{\text{gen}}$  suffices:

*T* is relatively *R*<sub>gen</sub>-terminating to *R* if every derivation *t*<sub>1</sub> → *t*<sub>2</sub>... for *t*<sub>1</sub> ∈ *T* contains finitely many →<sub>*R*</sub> steps

## Theorem (termination of narrowing via termination of rewriting)

Let  $\mathcal{R}$  be a left-linear constructor TRS T is  $\sim_{\mathcal{R}}$ -terminating  $\stackrel{\text{iff}}{\widehat{T}}$  if  $\widehat{T}$  is relatively  $\rightarrow_{\mathcal{R}_{gen}}$ -terminating to  $\mathcal{R}$  w.r.t. admissible derivations

## $\implies$ sufficient condition

# automating the termination analysis

#### The problem

Given  ${\mathcal R}$  and  ${\mathcal T}$ ,

 $\mathcal{T}$  is  $\sim_{\mathcal{R}}$ -terminating if  $\widehat{\mathcal{T}}$  is relatively  $\rightarrow_{\mathcal{R}_{gen}}$ -terminating to  $\mathcal{R}$ 

#### Drawback

• the set T is generally infinite

#### Solution: use abstract terms

- similar to modes in logic programming
- E.g.,  $\operatorname{add}(g, v)$  denotes the set of terms  $\operatorname{add}(t_1, t_2)$  with
  - t1 (definitely) ground
  - t<sub>2</sub> (possibly) variable
- concretization funcion  $\gamma$ ,

e.g.,  $\gamma(\operatorname{add}(g, v)) = \{\operatorname{add}(z, x), \operatorname{add}(z, z), \operatorname{add}(\operatorname{s}(z), x), \operatorname{add}(\operatorname{s}(z), z), \ldots\}$ 

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## The problem (revised)

Given  $\mathcal{R}$  and  $t^{lpha}$ ,  $\gamma(t^{lpha})$  is  $\sim_{\mathcal{R}}$ -terminating if  $\widehat{\gamma(t^{lpha})}$  is relatively  $\rightarrow_{\mathcal{R}_{gen}}$ -terminating to  $\mathcal{R}$ 

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 to filter away non-ground arguments of terms (equivalently, to filter away occurrences of gen)

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# Argument filterings [Kusakari, Nakamura, Toyama 1999]

 $ig(\pi(\mathsf{f})\subseteq\{1,\ldots,n\}$  for every defined function  $\mathsf{f}/nig)$ 

Argument filterings over terms & TRSs:

$$\pi(t) = \begin{cases} x & \text{if } t = x \\ c(\pi(t_1), \dots, \pi(t_n)) & \text{if } t = c(t_1, \dots, t_n) \\ f(\pi(t_{i_1}), \dots, \pi(t_{i_m})) & \text{if } t = f(t_1, \dots, t_n) \text{ and } \pi(f) = \{i_1, \dots, i_m\} \\ \pi(l \to r) = \pi(l) \to \pi(r) \end{cases}$$

From  $t^{\alpha}$  we infer a safe argument filtering  $\pi$  for  $t^{\alpha}$ •  $\pi(t^{\alpha}) = f(g, g, \dots, g)$ • for all  $s \rightsquigarrow t$ , if  $\pi(s|_{\rho})$  are ground then  $\pi(t|_q)$  are ground too

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## Proving termination automatically: approaches

## A direct approach

- based on dependency pairs [Arts, Giesl 2000]
- only a slight extension needed

#### A transformational approach

- based on argument filtering transformation [Kusakari, Nakamura, Toyama 1999]
- we use a simplified version (except for extra-variables)

Dependency pairs  $DP(\mathcal{R})$  of a TRS  $\mathcal{R}$ 

$$DP(\mathcal{R}) = \{ F(s_1, \ldots, s_n) \to G(t_1, \ldots, t_m) \mid f(s_1, \ldots, s_n) \to r \in \mathcal{R} \\ r|_{\rho} = g(t_1, \ldots, t_m) \}$$

where F, G are tuple symbols

#### Example

 $append(nil, y) \rightarrow y$   $append(cons(x, xs), y) \rightarrow cons(x, append(xs, y))$   $reverse(nil) \rightarrow nil$  $reverse(cons(x, xs)) \rightarrow append(reverse(xs), cons(x, nil))$ 

Germán Vidal (TU Valencia, Spain)

**Dependency pairs**  $DP(\mathcal{R})$  of a **TRS**  $\mathcal{R}$ 

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 $\begin{array}{l} \operatorname{append}(\operatorname{nil},y) \to y \\ \operatorname{append}(\operatorname{cons}(x,xs),y) \to \operatorname{cons}(x,\operatorname{append}(xs,y)) \\ \operatorname{reverse}(\operatorname{nil}) \to \operatorname{nil} \\ \operatorname{reverse}(\operatorname{cons}(x,xs)) \to \operatorname{append}(\operatorname{reverse}(xs),\operatorname{cons}(x,\operatorname{nil})) \end{array}$ 

APPEND(cons(x, xs), y)  $\rightarrow$  APPEND(xs, y)(1)REVERSE(cons(x, xs))  $\rightarrow$  REVERSE(xs)(2)REVERSE(cons(x, xs))  $\rightarrow$  APPEND(reverse(xs), cons(x, nil))(3)(3)(3)

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## Dependency pairs approach: differences

## Definition (chain)

A (possibly infinite) sequence of dependency pairs  $s_1 \rightarrow t_1$ ,  $s_2 \rightarrow t_2$ ,... from  $DP(\mathcal{R})$  is a  $(DP(\mathcal{R}), \mathcal{R}, \pi)$ -chain if

•  $\exists$  (constructor) substitution  $\sigma$  such that  $\widehat{t_i\sigma} \rightarrow^*_{\mathcal{R}_{gen}} \widehat{s_{i+1}\sigma}$  for  $i \ge 1$ 

•  $\pi(\widehat{s_i\sigma}), \pi(\widehat{t_i\sigma})$  contain no occurrences of gen

Three main extensions w.r.t. the standard notion:

- it is parameterized by  $\pi$
- $\bullet$  variables are replaced by gen and reductions w.r.t.  $\mathcal{R}_{gen}$
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## Example

Given the dependency pair

$$\mathsf{APPEND}(\mathsf{cons}(x, xs), y) \to \mathsf{APPEND}(xs, y) \quad (1)$$

we have an infinite  $(DP(\mathcal{R}), \mathcal{R}, \pi)$ -chain,  $(1), (1), \ldots$ , for

$$\pi(\mathsf{append}) = \pi(\mathsf{APPEND}) = \{2\}$$

since there exists  $\sigma = \{ y \mapsto \mathsf{nil} \}$  such that

$$\begin{array}{rcl} \mathsf{APPEND}(\mathsf{cons}(x, xs), y) & \to & \mathsf{APPEND}(xs, y) & (1) \\ & & & & & \downarrow^{\widehat{t\sigma}} \\ \mathsf{APPEND}(\mathsf{cons}(\mathsf{gen}, \mathsf{gen}), \mathsf{nil}) & \leftarrow & \mathsf{APPEND}(\mathsf{gen}, \mathsf{nil}) \end{array}$$

where  $\pi(\mathsf{APPEND}(\mathsf{gen},\mathsf{nil})) = \pi(\mathsf{APPEND}(\mathsf{cons}(\mathsf{gen},\mathsf{gen}),\mathsf{nil})) \in \mathcal{T}(\mathcal{F})$ 

(not a chain in the standard framework of rewriting)

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#### Theorem

Let  $\pi$  be a safe argument filtering for  $t^{\alpha}$  in  $\mathcal{R}$ If there is no infinite  $(DP(\mathcal{R}), \mathcal{R}, \pi)$ -chain, then  $\gamma(t^{\alpha})$  is  $\rightsquigarrow_{\mathcal{R}}$ -terminating

Now, we could adapt the processors of the dependency pair framework...

#### Argument filtering processor

E.g., we prove the soundness of transforming the DP problem

$$(DP(\mathcal{R}), \mathcal{R}, \pi) \implies (\pi(DP(\mathcal{R})), \pi(\mathcal{R}), id)$$

where  $id(f) = \{1, ..., n\}$  for all f/n occurring in  $\pi(\mathcal{R})$ 

 $(\pi(DP(\mathcal{R})), \pi(\mathcal{R}), id)$  is a standard DP problem, therefore,

• all DP processors [GTSKF06] for proving the termination of rewriting can be used for proving the termination of narrowing

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## Example

 $t^{lpha} = \operatorname{append}(g, v)$  $\pi = \{\operatorname{append} \mapsto \{1\}, \text{ reverse} \mapsto \{1\}\}$  ( $\pi$  is safe for  $t^{lpha}$ )

The argument filtering processor returns:

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Luckily, some extra-variables can be safely ignored...

- If the argument filtering is safe, extra-variables may only appear above the maximal function calls of the right-hand sides (thus π(DP(R)) never contains extra-variables)
- As for π(R), it should preserve the chains of dependency pairs:
   if s<sub>1</sub> → t<sub>1</sub>, s<sub>2</sub> → t<sub>2</sub>, ... is a chain in R
   then π(s<sub>1</sub>) → π(t<sub>1</sub>), π(s<sub>2</sub>) → π(t<sub>2</sub>), ... should be a chain in π(R)

• For this purpose, it suffices to consider extra-vars in those functions

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# A transformational approach

#### Our aim

- $\bullet$  transform the original TRS  ${\cal R}$  into a new TRS  ${\cal R}'$
- narrowing terminates in  ${\mathcal R}$  if rewriting terminates in  ${\mathcal R}'$

## Hence any termination technique for rewrite systems can be used to prove the termination of narrowing

Our transformation is a simplification of the *argument filtering transformation* (AFT) of [Kusakari, Nakamura, Toyama 1999]

## The transformation $\mathsf{AFT}_\pi(\mathcal{R})$

for every rule  $l \rightarrow r$  of the original rewrite system, produce

- a filtered rule  $\pi(l) \rightarrow \pi(r)$  and
- an additional rule  $\pi(l) \to \pi(t)$ , for each subterm t of r that is filtered away in  $\pi(r)$  and such that  $\pi(t)$  is not a constructor term.

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## Main result

#### Theorem

Let  $\pi$  be a safe argument filtering for  $t^{\alpha}$  in  $\mathcal{R}$  $\gamma(t^{\alpha})$  is  $\rightsquigarrow_{\mathcal{R}}$ -terminating if  $\mathsf{AFT}_{\pi}(\mathcal{R})$  is terminating

#### Therefore,

 AFT<sub>π</sub>(R) can be analyzed using standard techniques and tools for proving the termination of TRSs (no data generator is involved in the derivations of AFT<sub>π</sub>(R))

## Example

$$append(nil, y) \rightarrow y$$
  
 $append(cons(x, xs), y) \rightarrow cons(x, append(xs, y))$   
 $reverse(nil) \rightarrow nil$   
 $reverse(cons(x, xs)) \rightarrow append(reverse(xs), cons(x, nil))$ 

$$t^{lpha} = \operatorname{append}(g, v)$$
  
 $\pi = \{\operatorname{append} \mapsto \{1\}, \text{ reverse} \mapsto \{1\}\}$  ( $\pi$  is safe for  $t^{lpha}$ )

The transformation  $AFT_{\pi}(\mathcal{R})$  returns

$$\begin{array}{l} \operatorname{append(nil)} \to y & (y \text{ is an extra variable}) \\ \operatorname{append(cons}(x, xs)) \to \operatorname{cons}(x, \operatorname{append}(xs)) \\ \operatorname{reverse(nil)} \to \operatorname{nil} \\ \operatorname{reverse(cons}(x, xs)) \to \operatorname{append(reverse}(xs)) \end{array}$$

which is clearly **not** terminating

## Example

$$append(nil, y) \rightarrow y$$
  
 $append(cons(x, xs), y) \rightarrow cons(x, append(xs, y))$   
 $reverse(nil) \rightarrow nil$   
 $reverse(cons(x, xs)) \rightarrow append(reverse(xs), cons(x, nil))$ 

$$t^{lpha} = \operatorname{append}(g, v)$$
  
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The transformation  $AFT_{\pi}(\mathcal{R})$  returns

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which is clearly not terminating

# the technique in practice

3

## The termination tool TNT

It takes as input

- a left-linear constructor TRS
- an abstract term

and proceeds as follows:

- infers a safe argument filtering for the abstract term (a binding-time analysis)
- returns a transformed TRS using  $AFT_{\pi}$

Website: http://german.dsic.upv.es/filtering.html

The termination of the transformed TRS can be checked with APROVE

# [DEMO]

## Inference of safe argument filterings

We have adapted a simple (binding-time analysis)

• binding-times: definitively ground / possibly variable

$$g \sqcup g = g \qquad g \sqcup v = v \qquad v \sqcup g = v \qquad v \sqcup v = v$$
$$(g, v, g) \sqcup (g, g, v) = (g, v, v)$$
$$\{f \mapsto (g, v), g \mapsto (g, v)\} \sqcup \{f \mapsto (g, g), g \mapsto (v, g)\}$$
$$= \{f \mapsto (g, v), g \mapsto (v, v)\}$$

- binding-time environment: a substitution mapping variables to binding-times
- **<u>division</u>**: a mapping  $f/n \mapsto (m_1, \ldots, m_n)$  for every defined function, where each  $m_i$  is a binding-time

#### Auxiliary functions

$$B_{v}[[x]] g/n \rho = (\overbrace{g, \dots, g}^{n \text{ times}}) \qquad (\text{if } x \in \mathcal{V})$$

$$B_{v}[[c(t_{1}, \dots, t_{n})]] g/n \rho = B_{v}[[t_{1}]] g/n \rho \sqcup \dots \sqcup B_{v}[[t_{n}]] g/n \rho \qquad (\text{if } c \in \mathcal{C})$$

$$B_{v}[[f(t_{1}, \dots, t_{n})]] g/n \rho = bt \sqcup (B_{e}[[t_{1}]] \rho, \dots, B_{e}[[t_{n}]] \rho) \qquad (\text{if } f = g, f \in \mathcal{D})$$

$$bt \qquad (\text{if } f \neq g, f \in \mathcal{D})$$

$$where \ bt = B_{v}[[t_{1}]] g/n \rho \sqcup \dots \sqcup B_{v}[[t_{n}]] g/n \rho$$

$$B_{e}[[x]] \rho \qquad = x\rho \qquad (\text{if } x \in \mathcal{V})$$

 $B_e[[h(t_1,\ldots,t_n)]] \rho = B_e[[t_1]] \rho \sqcup \ldots \sqcup B_e[[t_n]] \rho \qquad (\text{if } h \in \mathcal{C} \cup \mathcal{D})$ 

#### Roughly speaking,

- (B<sub>v</sub>[[t]] g/n ρ) returns a sequence of n binding-times that denote the (lub of the) binding-times of the arguments of the calls to g/n that occur in t in the context of the binding-time environment ρ
- (B<sub>e</sub>[[t]] ρ) then returns g if t contains no variable which is bound to v in ρ, and v otherwise

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## **BTA** algorithm

Given an abstract term  $f_1(m_1, \ldots, m_{n_1})$ , the initial division is

 $div_0 = \{f_1 \mapsto (m_1, \dots, m_{n_1}), f_2 \mapsto (g, \dots, g), \dots, f_k \mapsto (g, \dots, g)\}$ where  $f_1/n_1, \dots, f_k/n_k$  are the defined functions of the TRS

**Iterative process** 

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where  $f_1/n_1, \ldots, f_k/n_k$  are the defined functions of the TRS

Iterative process

When  $div_i = div_{i+1}$  (fixpoint), the corresponding safe argument filtering  $\pi$  is obtained as follows:

Given the division

$$\left\{ div = \{ \mathsf{f}_1 \mapsto (m_1^1, \ldots, m_{n_1}^1), \ldots, \mathsf{f}_k \mapsto (m_1^k, \ldots, m_{n_k}^k) \} 
ight\}$$

we have

$$\left(\pi(\mathit{div}) = \{\mathsf{f}_1 \mapsto \{i \mid m_i^1 = g\}, \dots, \mathsf{f}_k \mapsto \{i \mid m_i^k = g\}\}\right)$$

 $\pi(div)$  is a safe argument filtering since the computed division div is congruent [JGS93]

(4) (E) (A) (E) (A)

Image: Image:

## Example

$$\begin{array}{rcl} \mathsf{mult}(\mathsf{z},y) & \to & \mathsf{z} & & \mathsf{add}(\mathsf{z},y) & \to & y \\ \mathsf{mult}(\mathsf{s}(x),y) & \to & \mathsf{add}(\mathsf{mult}(x,y),y) & & \mathsf{add}(\mathsf{s}(x),y) & \to & \mathsf{s}(\mathsf{add}(x,y)) \end{array}$$

Given the abstract term mult(g, v), the associated initial division is

$$\mathit{div}_0 = \{\mathsf{mult} \mapsto (g, v), \; \mathsf{add} \mapsto (g, g)\}$$

The next division,  $div_1$ , is obtained from the following expression:

$$\begin{array}{rcl} \operatorname{div}_1 = \{ \operatorname{mult} \mapsto (g, v) & \sqcup & B_v[[z]] \ \operatorname{mult}/2 \ \{y \mapsto v\} \\ & \sqcup & B_v[[\operatorname{add}(\operatorname{mult}(x, y), y)]] \ \operatorname{mult}/2 \ \{x \mapsto g, \ y \mapsto v\} \\ & \sqcup & B_v[[y]] \ \operatorname{mult}/2 \ \{y \mapsto g\} \\ & \sqcup & B_v[[s(\operatorname{add}(x, y))]] \ \operatorname{mult}/2 \ \{x \mapsto g, \ y \mapsto g\}, \\ & \operatorname{add} \mapsto (g, g) & \sqcup & B_v[[z]] \ \operatorname{add}/2 \ \{y \mapsto v\} \\ & \sqcup & B_v[[\operatorname{add}(\operatorname{mult}(x, y), y)]] \ \operatorname{add}/2 \ \{x \mapsto g, \ y \mapsto v\} \\ & \sqcup & B_v[[s(\operatorname{add}(x, y))]] \ \operatorname{add}/2 \ \{x \mapsto g, \ y \mapsto g\}. \end{array}$$

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## Example

$$\begin{array}{rrrr} \mathsf{mult}(z,y) & \to & z & & \mathsf{add}(z,y) & \to & y \\ \mathsf{mult}(\mathsf{s}(x),y) & \to & \mathsf{add}(\mathsf{mult}(x,y),y) & & \mathsf{add}(\mathsf{s}(x),y) & \to & \mathsf{s}(\mathsf{add}(x,y)) \end{array}$$

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# Example (cont'd)

Therefore, by evaluating the calls to  $B_v$ , we get

$$div_1 = \{ \mathsf{mult} \mapsto (g, v), \mathsf{add} \mapsto (v, v) \}$$

Note that the change in the binding-times of add comes from the evaluation of

$$B_{v}[[\operatorname{add}(\operatorname{mult}(x,y),y)]] \operatorname{add}/2 \{x \mapsto g, y \mapsto v\}$$

where a call to add appears (and every argument contains at least one possibly unknown value)

 $\Rightarrow$  If we compute  $div_2$  we get  $div_1 = div_2 \Longrightarrow div_1$  is a fixpoint

From this division, the associated safe argument filtering is

$$igl(\pi=\{ \mathsf{ mult}\mapsto \{1\}, \mathsf{ add}\mapsto \{\ \}\ igr)$$

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## Some refinements

#### Multiple abstract terms

Consider, e.g.,

$$eq(z, z) \rightarrow true$$
  
 $eq(s(x), s(y)) \rightarrow eq(x, y)$ 

and the set

$$T^{lpha} = \{ \mathsf{eq}(g, v), \; \mathsf{eq}(v, g) \}$$

Here, starting from

$$div_0 = \{ eq \mapsto (g, v) \sqcup (v, g) \} = \{ eq \mapsto (v, v) \}$$

is not a good idea ...

Solution

#### Lemma

Let  $\mathcal{R}$  be a TRS and  $T^{\alpha}$  be a finite set of abstract terms.  $\gamma(T^{\alpha})$  is  $\sim_{\mathcal{R}}$ -terminating for all  $t^{\alpha} \in T^{\alpha}$ .

Germán Vidal (TU Valencia, Spain)

Termination of Narrowing

FDI, U.C. Madrid 2009

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Germán Vidal (TU Valencia, Spain)

Termination of Narrowing

#### Non well-moded programs

Consider, e.g., 
$$\operatorname{eq}(z,z) \to \operatorname{true} = \operatorname{eq}(\operatorname{s}(x),\operatorname{s}(y)) \to \operatorname{eq}(y,x)$$
  
If we start with  $\operatorname{eq}(g,v)$ 

the only safe argument filtering is

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#### Non well-moded programs

lf

Consider, e.g.,  

$$eq(z,z) \rightarrow true$$
  
 $eq(s(x),s(y)) \rightarrow eq(y,x)$   
If we start with  
 $eq(g,v)$ 

the only safe argument filtering is

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$$eq_{gg}(z, z) \rightarrow true \qquad eq_{gv}(z, z) \rightarrow true$$

$$eq_{gg}(s(x), s(y)) \rightarrow eq_{gg}(y, x) \qquad eq_{gv}(s(x), s(y)) \rightarrow eq_{vg}(y, x)$$

$$eq_{vg}(z, z) \rightarrow true \qquad eq_{vv}(z, z) \rightarrow true$$

$$eq_{vg}(s(x), s(y)) \rightarrow eq_{gv}(y, x) \qquad eq_{vv}(s(x), s(y)) \rightarrow eq_{vv}(y, x)$$

Consider, e.g.,	а	$\rightarrow$	а
	b	$\rightarrow$	с
	с	$\rightarrow$	d

Although narrowing terminates for the abstract term b we get the argument filtering

$$\pi = \{\mathsf{a} \mapsto \{ \}, \ \mathsf{b} \mapsto \{ \}, \ \mathsf{c} \mapsto \{ \} \}$$

and then we fail to prove its termination...

## Solution

Remove function definitions not reachable from b (i.e., a ~
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# related work and conclusions

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## Related work

Schneider-Kamp *et al* [SKGST07] presented an automated termination analysis for logic programs:

- logic programs are first translated to TRSs
- logic variables are simulated by infinite terms

Main differences:

- data generators (reuse of results relating narrowing and rewriting)
- no transformational approach in [SKGST07]

Nishida et al [NSS03, NM06] adapted the dependency pair method for proving the termination of narrowing:

- direct approach (not based on using generators & rewriting)
- allow extra variables in TRSs and do not consider initial terms
- do not remove some (unnecessary) extra-variables (as we do)

Alpuente, Escobar, and Iborra [AEI08]

• extend the use of dependency pairs to narrowing over arbitrary TRS

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- new techniques for proving the termination of narrowing in left-linear constructor systems
- good potential for reusing existing techniques and tools for rewriting
- first tool for proving the termination of narrowing

**Future work** 

- extension to deal with extra-variables
- improve accuracy
- consider strategies (e.g., termination of lazy narrowing)

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