Termination of Narrowing in Left-Linear Constructor Systems*

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Abstract. Narrowing extends rewriting with logic capabilities by allowing logic variables in terms and replacing matching with unification. Narrowing has been widely used in different contexts, ranging from theorem proving to language design. Surprisingly, the termination of narrowing has been mostly overlooked. In this paper, we present a new approach for analyzing the termination of narrowing in left-linear constructor systems—a widely accepted class of systems—that allows us to reuse existing methods in the literature on termination of rewriting.

1 Introduction

The narrowing principle [35] generalizes term rewriting by allowing logic variables in terms—as in logic programming—and by replacing pattern matching with unification in order to (non-deterministically) reduce them. Unrestricted narrowing (i.e., not following any particular strategy for selecting reducible expressions) may have a huge—often infinite—search space, mainly because one can freely select any reducible expression *and* applicable rewrite rule at each narrowing step. Narrowing, originally introduced as an E-unification mechanism in equational theories, has been mostly used as the operational semantics of so called *functional logic* programming languages [21]. Recent examples of such languages based on narrowing are Curry [15] and Toy [27]. Currently, narrowing is regaining popularity in a number of other areas, like protocol verification [16, 28], model checking [17], partial evaluation [1, 32], refining methods for proving the termination of rewriting [8], type checking in the language Ωmega [34], etc.

Termination is a fundamental problem in term rewriting, as witnessed by the extensive literature on the subject (see, e.g., [13] and references therein). Surprisingly, the termination of narrowing has been mostly overlooked so far. To the best of our knowledge, no software tool for proving the termination of narrowing has ever been developed. Indeed, only a few approaches to this subject can be found in the literature (see a detailed account in Sect. 6).

In this work, we introduce a new approach to analyze the termination of narrowing by reusing existing results and tools for analyzing the termination of rewriting. The key idea is to consider variables as *data generators* in the context

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of rewriting. This means that one can analyze the termination of narrowing for the term $\mathsf{add}(x, \mathsf{z})$, where add is a defined function, x is a logic variable, and z is a constructor constant, by analyzing the termination of *rewriting* for all terms of the form $\mathsf{add}(t, \mathsf{z})$, where t stands for an arbitrary—possibly infinite—term. Intuitively speaking, we want t to take any possible value that could be computed by narrowing for the logic variable x in any derivation issuing from $\mathsf{add}(x, \mathsf{z})$, even if it goes on infinitely.

This relation between logic variables and (possibly infinite) terms has been recently exploited in order to eliminate logic variables from functional logic computations [6, 12]. A similar idea is also used in the termination analysis for logic programs of [33], where logic programs are transformed to rewrite systems and logic variables are then replaced with infinite terms (see Sect. 6).

Since data generators are, by definition, nonterminating, we introduce the use of *argument filterings* in Sect. 4 in order to filter away these data generators in rewrite derivations. Essentially, we consider two alternative approaches:

- The first technique is based on the well-known dependency pair framework [8, 20] for proving the termination of rewriting. We will show that only some slight modifications are required in order to be applicable in our setting.
- The second technique is based on the argument filtering transformation of Kusakari *et al* [26] and, given a TRS $\mathcal{R}$, produces a new rewrite system $\mathcal{R}'$ so that the termination of rewriting in $\mathcal{R}'$ implies the termination of narrowing in $\mathcal{R}$. Therefore, any method or termination tool for rewrite systems can be applied to $\mathcal{R}'$ in order to prove the termination of narrowing in $\mathcal{R}$.

Section 5 presents a technique for inferring appropriate argument filterings and reports on a prototype implementation of a termination tool, TNT, that follows the second approach above. First, the user introduces a rewrite system and an *abstract call* indicating the entry function to the program. The tool computes an argument filtering from the abstract call and, then, transforms the input system using this argument filtering. The termination of the transformed system is currently checked by using the AProVE tool [19].

The main contributions of this work can be summarized as follows: i) we introduce a sufficient and necessary condition for the termination of narrowing in left-linear constructor systems, a widely accepted class of systems; ii) we introduce two alternative approaches for analyzing the termination of narrowing w.r.t. a given argument filtering; and iii) we present an automatic tool for proving the termination of narrowing.

Finally, Sect. 6 includes a comparison to related work and Sect. 7 concludes. More details and proofs of all technical results can be found in [36].

2 Preliminaries

We assume familiarity with basic concepts of term rewriting and narrowing. We refer the reader to, e.g., [9] and [21] for further details.

Terms and Substitutions. A *signature* $\mathcal{F}$ is a set of function symbols. We often write $\mathsf{f}/n \in \mathcal{F}$ to denote that the arity of function f is n. Given a set of variables $\mathcal{V}$ with $\mathcal{F} \cap \mathcal{V} = \varnothing$, we denote the domain of *terms* by $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We assume that $\mathcal{F}$ always contains at least one constant $\mathsf{f}/0$. We use $\mathsf{f}, \mathsf{g}, \ldots$ to denote functions and $x, y, \ldots$ to denote variables. A *position* p in a term t is represented by a finite sequence of natural numbers, where ϵ denotes the root position. Positions are used to address the nodes of a term viewed as a tree. The root symbol of a term t is denoted by $\mathsf{root}(t)$. We let $t|_p$ denote the *subterm* of t at position p and $t[s]_p$ the result of *replacing the subterm* $t|_p$ by the term s. $\mathcal{V}\mathsf{ar}(t)$ denotes the set of variables appearing in t. A term t is *ground* if $\mathcal{V}\mathsf{ar}(t) = \varnothing$. We write $\mathcal{T}(\mathcal{F})$ as a shorthand for the set of ground terms $\mathcal{T}(\mathcal{F}, \varnothing)$.

A *substitution* $\sigma : \mathcal{V} \mapsto \mathcal{T}(\mathcal{F}, \mathcal{V})$ is a mapping from variables to terms such that $\mathcal{D}\mathsf{om}(\sigma) = \{x \in \mathcal{V} \mid x \neq \sigma(x)\}$ is its domain. The set of variables introduced by a substitution σ is denoted by $\mathcal{R}\mathsf{an}(\sigma) = \cup_{x \in \mathcal{D}\mathsf{om}(\sigma)} \mathcal{V}\mathsf{ar}(x\sigma)$. Substitutions are extended to morphisms from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ in the natural way. We denote the application of a substitution σ to a term t by $t\sigma$ (rather than $\sigma(t)$). The identity substitution is denoted by *id*. A *variable renaming* is a substitution that is a bijection on $\mathcal{V}$. A substitution σ is *more general* than a substitution θ, denoted by $\sigma \leqslant \theta$, if there is a substitution δ such that $\delta \circ \sigma = \theta$, where "$\circ$" denotes the composition of substitutions (i.e., $\sigma \circ \theta(x) = x\theta\sigma$). The *restriction* $\theta\restriction_V$ of a substitution θ to a set of variables V is defined as follows: $x\theta\restriction_V = x\theta$ if $x \in V$ and $x\theta\restriction_V = x$ otherwise. We say that $\theta = \sigma\ [V]$ if $\theta\restriction_V = \sigma\restriction_V$.

A term t_2 is an *instance* of a term t_1 (or, equivalently, t_1 is *more general* than t_2), in symbols $t_1 \leqslant t_2$, if there is a substitution σ with $t_2 = t_1\sigma$. Two terms t_1 and t_2 are *variants* (or equal up to variable renaming) if $t_1 = t_2\rho$ for some variable renaming ρ. A *unifier* of two terms t_1 and t_2 is a substitution σ with $t_1\sigma = t_2\sigma$; furthermore, σ is the *most general unifier* of t_1 and t_2, denoted by $\mathsf{mgu}(t_1, t_2)$ if, for every other unifier θ of t_1 and t_2, we have that $\sigma \leqslant \theta$.

TRSs and Rewriting. A set of rewrite rules $l \to r$ such that l is a nonvariable term and r is a term whose variables appear in l is called a *term rewriting system* (TRS for short); terms l and r are called the left-hand side and the right-hand side of the rule, respectively. We restrict ourselves to finite signatures and TRSs. Given a TRS $\mathcal{R}$ over a signature $\mathcal{F}$, the *defined* symbols $\mathcal{D}$ are the root symbols of the left-hand sides of the rules and the *constructors* are $\mathcal{C} = \mathcal{F} \setminus \mathcal{D}$.

We use the notation $\mathcal{F} = \mathcal{D} \uplus \mathcal{C}$ to point out that $\mathcal{D}$ are the defined function symbols and $\mathcal{C}$ are the constructors of a signature $\mathcal{F}$, with $\mathcal{D} \cap \mathcal{C} = \varnothing$. The domains $\mathcal{T}(\mathcal{C}, \mathcal{V})$ and $\mathcal{T}(\mathcal{C})$ denote the sets of *constructor terms* and *ground constructor terms*, respectively. A substitution σ is (ground) *constructor*, if $x\sigma$ is a (ground) constructor term for all $x \in \mathcal{D}\mathsf{om}(\sigma)$.

A TRS $\mathcal{R}$ is a *constructor system* if the left-hand sides of its rules have the form $\mathsf{f}(s_1, \ldots, s_n)$ where s_i are constructor terms, i.e., $s_i \in \mathcal{T}(\mathcal{C}, \mathcal{V})$, for all $i = 1, \ldots, n$. A term t is *linear* if every variable of $\mathcal{V}$ occurs at most once in t. A TRS $\mathcal{R}$ is *left-linear* if l is linear for every rule $l \to r \in \mathcal{R}$.

For a TRS $\mathcal{R}$, we define the associated rewrite relation $\to_{\mathcal{R}}$ as follows: given terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, we have $s \to_{\mathcal{R}} t$ iff there exists a position p in s, a rewrite rule $l \to r \in \mathcal{R}$ and a substitution σ with $s|_p = l\sigma$ and $t = s[r\sigma]_p$; the rewrite step is often denoted by $s \to_{p,l\to r} t$ to make explicit the position and rule used in this step. The instantiated left-hand side $l\sigma$ is called a *redex*.

A term t is called *irreducible* or in *normal form* in a TRS $\mathcal{R}$ if there is no term s with $t \to_{\mathcal{R}} s$. A *derivation* is a (possibly empty) sequence of rewrite steps. Given a binary relation $\to$, we denote by $\to^+$ the transitive closure of $\to$ and by $\to^*$ its reflexive and transitive closure. Thus $t \to^*_{\mathcal{R}} s$ means that t can be reduced to s in $\mathcal{R}$ in zero or more steps; we also use $t \to^n_{\mathcal{R}} s$ to denote that t can be reduced to s in exactly n rewrite steps.

Narrowing. The *narrowing* principle [35] mainly extends term rewriting by replacing pattern matching with unification, so that terms containing logic variables can also be reduced by non-deterministically instantiating these variables. Formally, given a TRS $\mathcal{R}$ and two terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, we have that $s \leadsto_{\mathcal{R}} t$ is a *narrowing step* iff there exist[1]

- a nonvariable position p of s,
- a variant $R = (l \to r)$ of a rule in $\mathcal{R}$,
- a substitution $\sigma = \mathsf{mgu}(s|_p, l)$ which is the most general unifier of $s|_p$ and l,

and $t = (s[r]_p)\sigma$. We often write $s \leadsto_{p,R,\theta} t$ (or simply $s \leadsto_\theta t$) to make explicit the position, rule, and substitution of the narrowing step, where $\theta = \sigma\restriction_{\mathcal{V}\mathsf{ar}(s)}$ (i.e., we label the narrowing step only with the bindings for the narrowed term). A *narrowing derivation* $t_0 \leadsto^*_\sigma t_n$ denotes a sequence of narrowing steps $t_0 \leadsto_{\sigma_1} \ldots \leadsto_{\sigma_n} t_n$ with $\sigma = \sigma_n \circ \cdots \circ \sigma_1$ (if $n = 0$ then $\sigma = id$). Given a narrowing derivation $s \leadsto^*_\sigma t$, we say that σ is a computed *answer* for s.

Example 1. Consider the following TRS $\mathcal{R}$ defining the addition $\mathsf{add}/2$ on natural numbers built from $\mathsf{z}/0$ and $\mathsf{s}/1$:

$$\begin{array}{rcll} \mathsf{add}(\mathsf{z}, y) & \to & y & (R_1) \\ \mathsf{add}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{add}(x, y)) & (R_2) \end{array}$$

Given the term $\mathsf{add}(x, \mathsf{s}(\mathsf{z}))$, we have infinitely many narrowing derivations issuing from $\mathsf{add}(x, \mathsf{s}(\mathsf{z}))$, e.g.

$$\begin{array}{l} \mathsf{add}(x, \mathsf{s}(\mathsf{z})) \leadsto_{\epsilon,R_1,\{x\mapsto \mathsf{z}\}} \mathsf{s}(\mathsf{z}) \\ \mathsf{add}(x, \mathsf{s}(\mathsf{z})) \leadsto_{\epsilon,R_2,\{x\mapsto \mathsf{s}(y_1)\}} \mathsf{s}(\mathsf{add}(y_1, \mathsf{s}(\mathsf{z}))) \leadsto_{1,R_1,\{y_1\mapsto \mathsf{z}\}} \mathsf{s}(\mathsf{s}(\mathsf{z})) \\ \ldots \end{array}$$

with computed answers $\{x \mapsto \mathsf{z}\}$, $\{x \mapsto \mathsf{s}(\mathsf{z})\}$, etc.

[1] We consider the so called *most general* narrowing, i.e., the mgu of the selected subterm and the left-hand side of a rule—rather than an ordinary unifier—is computed at each narrowing step.

3 Termination of Narrowing via Termination of Rewriting

We first introduce our notion of termination, which is parameterized by a given binary relation:

Definition 1 (termination). *Let T be a set of terms. Given a binary relation $\propto$ on terms, we say that T is $\propto$-terminating iff there is no term $t_1 \in T$ such that there exists an infinite sequence of the form $t_1 \propto t_2 \propto t_3 \propto \ldots$*

We say that a term t is $\propto$-terminating iff the set $\{t\}$ is $\propto$-terminating.

The usual notion of termination can then be formulated as follows: a TRS is *terminating* iff $\mathcal{T}(\mathcal{F})$ is $\to_{\mathcal{R}}$-terminating. As for narrowing, we say that a TRS $\mathcal{R}$ is terminating *w.r.t. narrowing* iff $\mathcal{T}(\mathcal{F},\mathcal{V})$ is $\leadsto_{\mathcal{R}}$-terminating.

In general, however, only rather trivial TRSs are terminating w.r.t. narrowing. Consider, for instance, the following TRS $\mathcal{R} = \{\mathsf{f}(\mathsf{s}(x), y) \to \mathsf{f}(x, y)\}$. Although every term of the form $\mathsf{f}(t_1, t_2)$ has a finite rewrite derivation, we can easily find a term, e.g., $\mathsf{f}(w, \mathsf{z})$, such that an infinite narrowing derivation exists:

$$\mathsf{f}(w,\mathsf{z}) \leadsto_{\{w \mapsto \mathsf{s}(x_1)\}} \mathsf{f}(x_1,\mathsf{z}) \leadsto_{\{x_1 \mapsto \mathsf{s}(x_2)\}} \mathsf{f}(x_2,\mathsf{z}) \leadsto_{\{x_2 \mapsto \mathsf{s}(x_3)\}} \ldots$$

Therefore, we focus on the termination of narrowing w.r.t. a *given set of terms*, which explains our formulation of termination in Def. 1 above.

The following result provides a first—sufficient but not necessary—condition for the termination of narrowing in terms of the termination of rewriting.

Theorem 1. *Let $\mathcal{R}$ be a TRS and T be a finite set of terms. Let $T^* = \{t\sigma \mid t \in T \text{ and } t \leadsto_\sigma^* s \text{ in } \mathcal{R}\}$. T is $\leadsto_{\mathcal{R}}$-terminating if T^* is finite (modulo variable renaming) and $\to_{\mathcal{R}}$-terminating.*

The following example illustrates why the above condition is not necessary:

Example 2. Consider the following TRS: $\mathcal{R} = \{\mathsf{f}(\mathsf{a}) \to \mathsf{b},\ \mathsf{a} \to \mathsf{a}\}$. Given the set of terms $T = \{\mathsf{f}(x)\}$, we have that T is $\leadsto_{\mathcal{R}}$-terminating since the only narrowing derivation is $\mathsf{f}(x) \leadsto_{\{x \mapsto \mathsf{a}\}} \mathsf{b}$. However, $T^* = \{\mathsf{f}(\mathsf{a})\}$ is finite but not $\to_{\mathcal{R}}$-terminating: $\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{a}) \to \ldots$

Verifying the finiteness and $\to_{\mathcal{R}}$-termination of T^* is generally, not only undecidable, but also rather difficult to approximate since one should approximate all possible narrowing derivations issuing from the terms in T. Therefore, we now introduce an alternative—easier to check—condition.

Firstly, we restrict ourselves to a narrowing strategy over a class of TRSs in which the terms introduced by instantiation cannot be narrowed (this will avoid, e.g., the situation of Ex. 2). Many useful narrowing strategies fulfill this condition, e.g., basic [23] and innermost basic narrowing [22] over arbitrary TRSs, lazy [29] and needed[2] narrowing [5] over left-linear constructor TRSs, etc. Actually, any narrowing strategy over left-linear constructor systems computes only constructor substitutions (a formal proof can be found in [36]).

[2] Although needed narrowing [5] does not compute mgu's (basically, some bindings are anticipated to ensure that all narrowing steps are *needed*), it computes constructor substitutions (see [3, Lemma 11]) and, thus, our forthcoming results also apply.

Secondly, as mentioned in the introduction, we regard variables in narrowing as *generators* of possibly infinite (constructor) terms from the point of view of rewriting. For this purpose, we introduce a fixed fresh function symbol "gen" which does not appear in the signature of any TRS. The following definition is a simplified version of the original notion of a *generator* in [6]:

Definition 2 (data generator, gen). *Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F} = \mathcal{D} \uplus \mathcal{C}$. We denote by $\mathcal{R}_{\mathsf{gen}}$ a TRS over $\mathcal{F} \uplus \{\mathsf{gen}\}$ resulting from augmenting $\mathcal{R}$ with the following set of rewrite rules:*

$$\{\mathsf{gen} \to \mathsf{c} \mid \mathsf{c}/0 \in \mathcal{C}\} \cup \{\mathsf{gen} \to \mathsf{c}(\overbrace{\mathsf{gen}, \ldots, \mathsf{gen}}^{n \text{ times}}) \mid \mathsf{c}/n \in \mathcal{C},\ n > 0\}$$

Example 3. For instance, for the TRS $\mathcal{R}$ of Ex. 1 with $\mathcal{C} = \{\mathsf{z}/0, \mathsf{s}/1\}$, we have $\mathcal{R}_{\mathsf{gen}} = \mathcal{R} \cup \{\mathsf{gen} \to \mathsf{z},\ \mathsf{gen} \to \mathsf{s}(\mathsf{gen})\}$.

Trivially, the function gen can be (non-deterministically) reduced to any ground constructor term. Variables are then replaced by generators in the obvious way:

Definition 3 (variable elimination, $\widehat{t}$, $\widehat{T}$). *Given a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ over a signature $\mathcal{F}$, we let $\widehat{t} = t\sigma$, with $\sigma = \{x \mapsto \mathsf{gen} \mid x \in \mathcal{V}\mathsf{ar}(t)\}$. Analogously, given a set of terms $T \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$, we let $\widehat{T} = \{\widehat{t} \mid t \in T\} \subseteq \mathcal{T}(\mathcal{F} \uplus \{\mathsf{gen}\})$.*

Note that $\widehat{t}$ is always ground for any given term t since all variables occurring in t are replaced by function gen.

Now, we state the correctness of the variable elimination, an easy consequence of the results in [6] (a complete proof can be found in [36]). Our first result shows that every narrowing computation can be mimicked by a rewrite derivation if logic variables are replaced with gen in the initial term:

Lemma 1 (completeness). *Let $\mathcal{R}$ be a left-linear constructor TRS over a signature $\mathcal{F} = \mathcal{D} \uplus \mathcal{C}$ and $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ be a term. If $s \leadsto_{p,R,\sigma} t$ in $\mathcal{R}$, then $\widehat{s} \to^* \widehat{s\sigma} \to_{p,R} \widehat{t}$ in $\mathcal{R}_{\mathsf{gen}}$.*

Unfortunately, variable elimination is not generally sound because repeated variables are bound to the same value in a narrowing computation, while different occurrences of gen, though arising from the replacement of the same variable, can be reduced to different terms:

Example 4. Consider again the TRS $\mathcal{R}$ of Ex. 1 and the term $t = \mathsf{add}(x, x)$. Clearly, it can only be narrowed to an even number: z, $\mathsf{s}(\mathsf{s}(\mathsf{z}))$, ... However, $\widehat{t}$ can also be reduced to an odd number, e.g., $\widehat{t} = \mathsf{add}(\mathsf{gen}, \mathsf{gen}) \to \mathsf{add}(\mathsf{z}, \mathsf{gen}) \to \mathsf{gen} \to \mathsf{s}(\mathsf{gen}) \to \mathsf{s}(\mathsf{z})$.

To avoid such derivations, the notion of *admissible* derivation [6] is introduced:

Definition 4 (admissible derivation). *Let $\mathcal{R}$ be a TRS over $\mathcal{F}$ and $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ be a term. A derivation for $\widehat{t}$ in $\mathcal{R}_{\mathsf{gen}}$ is called admissible iff all the occurrences of gen originating from the replacement of the same variable are reduced to the same term in this derivation.*

Now, we can already state the soundness of variable elimination:

Lemma 2 (soundness). *Let $\mathcal{R}$ be a left-linear constructor TRS over a signature $\mathcal{F} = \mathcal{D} \uplus \mathcal{C}$ and $s' \in \mathcal{T}(\mathcal{F} \cup \{\mathsf{gen}\}, \mathcal{V})$ be a term. If $s' \to^* s'' \to_{p,R} t'$ is an admissible derivation in $\mathcal{R}_{\mathsf{gen}}$ and $R \in \mathcal{R}$, then $s \rightsquigarrow^*_{\mathcal{R}} t$ with $\widehat{s} = s'$ and $\widehat{t\sigma} = t'$ for some constructor substitution σ.*

Obviously, given a TRS $\mathcal{R}$, no set of terms containing occurrences of gen is generally $\to_{\mathcal{R}_{\mathsf{gen}}}$-terminating because of the definition of function gen. Luckily, we are interested in a weaker property: we may allow infinite derivations in $\mathcal{R}_{\mathsf{gen}}$ as long as the number of functions different from gen reduced in these derivations is kept finite (i.e., gen is only used to produce the values needed to perform the next rewrite step). This idea is formalized by using the notion of *relative termination* [25]:

Definition 5 (relative termination). *Let $\mathcal{R}$ and $\mathcal{Q}$ be rewrite systems. Let T be a set of terms. T is relatively $\to_{\mathcal{R}\cup\mathcal{Q}}$-terminating to $\mathcal{R}$ if every infinite derivation $t_0 \to_{\mathcal{R}\cup\mathcal{Q}} t_1 \to_{\mathcal{R}\cup\mathcal{Q}} \ldots$ contains only finitely many $\to_{\mathcal{R}}$-steps.*

The following theorem states one of the main results of this paper:

Theorem 2. *Let $\mathcal{R}$ be a left-linear constructor TRS over a signature $\mathcal{F} = \mathcal{D} \uplus \mathcal{C}$ and let $T \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$ be a set of terms. Then, T is $\rightsquigarrow_{\mathcal{R}}$-terminating iff $\widehat{T}$ is relatively $\to_{\mathcal{R}_{\mathsf{gen}}}$-terminating to $\mathcal{R}$.*

The above result lays the ground for analyzing the termination of narrowing by reusing existing techniques for proving the termination of rewriting. The next section presents two such approaches.

4 Automating the Termination Analysis

4.1 From Abstract Terms to Argument Filterings

In general, we are not interested in providing a set of terms T for proving that T is $\rightsquigarrow$-terminating. Rather, it is much more convenient to allow the user to provide a higher-level specification of the function calls in which she is interested in. For this purpose, we introduce the notion of an *abstract term*, which is inspired by the *mode declarations* of logic programming.

Definition 6 (abstract term). *Let $\mathcal{F} = \mathcal{D} \uplus \mathcal{C}$ be a signature. An abstract term over $\mathcal{F}$ has the form $\mathsf{f}(m_1, \ldots, m_n)$, where $\mathsf{f} \in \mathcal{D}$ is a defined function symbol and m_i, $i = 1, \ldots, n$, is either g (definitely **g**round) or v (possibly **v**ariable).*

Any abstract term implicitly induces a (possibly infinite) set of terms:

Definition 7 (concretization, γ). *Let $\mathcal{F}$ be a signature and t^α an abstract term over $\mathcal{F}$. The concretization of t^α, in symbols $\gamma(t^\alpha)$, is obtained as follows:*

$$\gamma(\mathsf{f}(m_1, \ldots, m_n)) = \{\mathsf{f}(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \mid t_i \in \mathcal{T}(\mathcal{C}) \text{ if } m_i = g,\ i = 1, \ldots, n\}$$

Given a set of abstract terms T^α, we let $\gamma(T^\alpha) = \{\gamma(t^\alpha) \mid t^\alpha \in T^\alpha\}$.

Consider the TRS of Ex. 1 and the abstract term $\mathsf{add}(g, v)$. Then, $\gamma(\mathsf{add}(g, v)) = \{\mathsf{add}(\mathsf{z}, x), \mathsf{add}(\mathsf{z}, \mathsf{z}), \mathsf{add}(\mathsf{s}(\mathsf{z}), x), \mathsf{add}(\mathsf{s}(\mathsf{z}), \mathsf{z}), \mathsf{add}(\mathsf{s}(\mathsf{z}), \mathsf{s}(x)), \mathsf{add}(\mathsf{s}(\mathsf{z}), \mathsf{s}(\mathsf{z})), \ldots\}$.

Thanks to Theorem 2, given a set of abstract terms T^α, we can prove that $\gamma(T^\alpha)$ is $\rightsquigarrow_{\mathcal{R}}$-terminating by proving that $\widehat{\gamma(T^\alpha)}$ is relatively $\to_{\mathcal{R}_{\mathsf{gen}}}$-terminating to $\mathcal{R}$. This approach, however, presents two drawbacks:

- the set $\gamma(T^\alpha)$ is generally infinite and
- checking relative termination require non-standard techniques and tools.

In order to overcome these drawbacks, we introduce the use of (a simplified version of) argument filterings:

Definition 8 (argument filtering, π). *An argument filtering over a signature $\mathcal{F} = \mathcal{D} \uplus \mathcal{C}$ is a function π such that, for every defined function $\mathsf{f}/n \in \mathcal{D}$, we have $\pi(\mathsf{f}) \subseteq \{1, \ldots, n\}$. Argument filterings are extended to terms as follows:*[3]

- $\pi(x) = x$ *for all* $x \in \mathcal{V}$,
- $\pi(\mathsf{c}(t_1, \ldots, t_n)) = \mathsf{c}(\pi(t_1), \ldots, \pi(t_n))$ *for all* $\mathsf{c}/n \in \mathcal{C}$, $n \geq 0$, *and*
- $\pi(\mathsf{f}(t_1, \ldots, t_n)) = \mathsf{f}(\pi(t_{i_1}), \ldots, \pi(t_{i_m}))$ *for all* $\mathsf{f}/n \in \mathcal{F}$, $n \geq 0$, *where* $\pi(\mathsf{f}) = \{i_1, \ldots, i_m\}$ *and* $1 \leq i_1 < \ldots < i_m \leq n$.

Given a TRS $\mathcal{R}$, we let $\pi(\mathcal{R}) = \{\pi(l) \to \pi_{rhs}(r) \mid l \to r \in \mathcal{R}\}$, where the auxiliary function π_{rhs} is defined as follows:

- $\pi_{rhs}(x) = \bot$ *for all* $x \in \mathcal{V}$,
- $\pi_{rhs}(\mathsf{c}(t_1, \ldots, t_n)) = \mathsf{c}(\pi_{rhs}(t_1), \ldots, \pi_{rhs}(t_n))$ *for all* $\mathsf{c}/n \in \mathcal{C}$, $n \geq 0$, *and*
- $\pi_{rhs}(\mathsf{f}(t_1, \ldots, t_n)) = \mathsf{f}(\pi(t_{i_1}), \ldots, \pi(t_{i_m}))$ *for all* $\mathsf{f}/n \in \mathcal{F}$, $n \geq 0$, *where* $\pi(\mathsf{f}) = \{i_1, \ldots, i_m\}$ *and* $1 \leq i_1 < \ldots < i_m \leq n$

where $\bot$ is a fresh constant constructor not appearing in $\mathcal{C}$.

The original notion of *argument filtering* in [8, 26] may return a single argument position so that $\pi(\mathsf{f}(t_1, \ldots, t_n)) = \pi(t_i)$ if $\pi(\mathsf{f}) = i$; furthermore, it applies to both constructor and defined function symbols. We consider a simpler definition because our argument filterings will be automatically derived from a set of abstract terms (cf. Sect. 5), where only defined function symbols occur.

On the other hand, our argument filterings replace those variables of the right-hand sides that are not below a defined function symbol by a fresh constant $\bot$. This is done in order to avoid the introduction of *extra* variables (i.e., variables that appear in the right-hand side of a rule but not in its left-hand side). Consider, e.g., the rule $\mathsf{add}(\mathsf{z}, y) \to y$ and the argument filtering $\pi = \{\mathsf{add} \mapsto \{1\}\}$. Then, $(\pi(\mathsf{add}(\mathsf{z}, y)) \to \pi(y)) = (\mathsf{add}(\mathsf{z}) \to y)$ that contains an extra variable y. Our definition above returns instead $(\pi(\mathsf{add}(\mathsf{z}, y)) \to \pi_{rhs}(y)) = (\mathsf{add}(\mathsf{z}) \to \bot)$.

In the following, though, we are not interested in arbitrary argument filterings but only in what we call *safe* argument filterings.

[3] By abuse of notation, we keep the same symbol for the original function and the filtered function with a possibly different arity.

Definition 9 (safe argument filtering). *Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F} = \mathcal{D} \uplus \mathcal{C}$ and let T^α be a finite set of abstract terms. We say that an argument filtering π is safe for T^α in $\mathcal{R}$ iff*

- *for all $t^\alpha \in T^\alpha$, if $\pi(t^\alpha) = \mathsf{f}(m_1, \ldots, m_n)$, then $m_i = g$ for all $i = 1, \ldots, n$;*
- *for all narrowing step $s_1 \leadsto_{\mathcal{R}} s_2$, if $\pi(s_1|_p) \in \mathcal{T}(\mathcal{F})$ for all subterm $s_1|_p$ with $\mathsf{root}(s_1|_p) \in \mathcal{D}$, then $\pi(s_2|_q) \in \mathcal{T}(\mathcal{F})$ for all subterm $s_2|_q$ with $\mathsf{root}(s_2|_q) \in \mathcal{D}$.*

Intuitively speaking, an argument filtering π is safe for a set of abstract terms T^α if π filters away all non-ground arguments of the terms in $\gamma(T^\alpha)$ as well as the non-ground arguments of any function call that can be obtained by narrowing.

Example 5. Consider the TRS $\mathcal{R} = \{\mathsf{f}(\mathsf{s}(x), y) \to \mathsf{f}(y, x)\}$ and the set $T^\alpha = \{\mathsf{f}(g, v)\}$. Given the argument filtering $\pi = \{\mathsf{f} \mapsto \{1\}\}$, although $\pi(\mathsf{f}(g, v)) = \mathsf{f}(g)$ holds (the first condition in Def. 9), this argument filtering is not safe because there exists a narrowing step $\mathsf{f}(\mathsf{s}(\mathsf{z}), x) \leadsto \mathsf{f}(x, \mathsf{z})$ such that $\pi(\mathsf{f}(\mathsf{s}(\mathsf{z}), x)) = \mathsf{f}(\mathsf{s}(\mathsf{z}))$ is ground but $\pi(\mathsf{f}(x, \mathsf{z})) = \mathsf{f}(x)$ is not.

A useful property is that the filtered form of a TRS does not contain extra variables when the argument filtering is safe (see [36]).

In the following, we consider that the input for the termination analysis is a left-linear TRS together with a safe argument filtering. An algorithm for generating safe argument filterings from abstract terms can be found in Sect. 5.

4.2 A Direct Approach to Termination Analysis

In this section, we present a direct approach for proving the termination of narrowing by extending the well-known *dependency pair* technique [8].

The remainder of this section adapts and extends some of the developments in [8]. Given a TRS $\mathcal{R}$ over a signature $\mathcal{F}$, for each $\mathsf{f}/n \in \mathcal{F}$, we let $\mathsf{f}^\sharp/n$ be a fresh *tuple symbol* (a constructor); we often write F instead of $\mathsf{f}^\sharp$. Given a term $\mathsf{f}(t_1, \ldots, t_n)$ with $\mathsf{f} \in \mathcal{D}$, we let $t^\sharp$ denote $\mathsf{f}^\sharp(t_1, \ldots, t_n)$.

Definition 10 (dependency pair [8]). *Given a TRS $\mathcal{R}$ over a signature $\mathcal{F} = \mathcal{D} \uplus \mathcal{C}$, the associated set of dependency pairs, $DP(\mathcal{R})$, is defined as follows:*[4]

$$DP(\mathcal{R}) = \{l^\sharp \to t^\sharp \mid l \to r \in \mathcal{R},\ r|_p = t,\ \textit{and}\ \mathsf{root}(t) \in \mathcal{D}\}$$

Example 6. Consider the following TRS $\mathcal{R}$ defining the functions append and reverse over lists built from nil (the empty list) and cons:

$$\begin{aligned}
\mathsf{append}(\mathsf{nil}, y) &\to y \\
\mathsf{append}(\mathsf{cons}(x, xs), y) &\to \mathsf{cons}(x, \mathsf{append}(xs, y)) \\
\mathsf{reverse}(\mathsf{nil}) &\to \mathsf{nil} \\
\mathsf{reverse}(\mathsf{cons}(x, xs)) &\to \mathsf{append}(\mathsf{reverse}(xs), \mathsf{cons}(x, \mathsf{nil}))
\end{aligned}$$

[4] Note that if $\mathcal{R}$ is a TRS, so is $DP(\mathcal{R})$.

Here, we have the following dependency pairs $DP(\mathcal{R})$:

$$\begin{aligned}
&\mathsf{APPEND}(\mathsf{cons}(x, xs), y) \to \mathsf{APPEND}(xs, y) && (1)\\
&\mathsf{REVERSE}(\mathsf{cons}(x, xs)) \to \mathsf{REVERSE}(xs) && (2)\\
&\mathsf{REVERSE}(\mathsf{cons}(x, xs)) \to \mathsf{APPEND}(\mathsf{reverse}(xs), \mathsf{cons}(x, \mathsf{nil})) && (3)
\end{aligned}$$

In order to prove termination, we should try to prove that there are no infinite chains of dependency pairs. The standard notion of *chain* in [8], however, cannot be used because we are interested in the termination of narrowing (i.e., the relative termination of rewrite sequences in which variables are replaced by gen).

Definition 11 (chain). *Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F}$ and let π be an argument filtering over $\mathcal{F}$ that is extended over tuple symbols so that $\pi(f^\sharp) = \pi(f)$ for all $f \in \mathcal{D}$. A (possibly infinite) sequence of pairs $s_1 \to t_1$, $s_2 \to t_2$, ... from $DP(\mathcal{R})$ is a $(DP(\mathcal{R}), \mathcal{R}, \pi)$-chain if the following conditions hold:*[5]

- *there exists a constructor substitution σ such that $\widehat{t_i\sigma} \to^*_{\mathcal{R}_{\mathsf{gen}}} \widehat{s_{i+1}\sigma}$ for every two consecutive pairs in the sequence;*
- *we have $\pi(\widehat{s_i\sigma}), \pi(\widehat{t_i\sigma}) \in \mathcal{T}(\mathcal{F})$ for all $i > 0$ (i.e., π filters away all occurrences of gen).*

Example 7. Consider the TRS $\mathcal{R}$ of Example 6 and its dependency pairs $DP(\mathcal{R})$. Here, $\mathcal{R}_{\mathsf{gen}} = \mathcal{R} \cup \{\mathsf{gen} \to \mathsf{nil},\ \mathsf{gen} \to \mathsf{cons}(\mathsf{gen}, \mathsf{gen}),\ \mathsf{gen} \to \mathsf{z},\ \mathsf{gen} \mapsto \mathsf{s}(\mathsf{gen})\}$. Then, we have that "$(1), (1), \ldots$" is an infinite $(DP(\mathcal{R}), \mathcal{R}, \pi)$-chain for any argument filtering in which $\pi(\mathsf{APPEND}) = \{2\}$ since there exists a substitution $\sigma = \{y \mapsto \mathsf{nil}\}$ such that (we denote the dependency pair (1) by $l_1 \to t_1$)

$$\widehat{t_1\sigma} = \mathsf{APPEND}(\mathsf{gen}, \mathsf{nil}) \to_{\mathcal{R}_{\mathsf{gen}}} \mathsf{APPEND}(\mathsf{cons}(\mathsf{gen}, \mathsf{gen}), \mathsf{nil}) = \widehat{l_1\sigma}$$

and $\pi(\mathsf{APPEND}(\mathsf{gen}, \mathsf{nil})) = \pi(\mathsf{APPEND}(\mathsf{cons}(\mathsf{gen}, \mathsf{gen}), \mathsf{nil})) = \mathsf{nil} \in \mathcal{T}(\mathcal{F})$. Note that it would be not a chain in the standard dependency pair method.

The following result states the soundness of our approach:

Theorem 3. *Let $\mathcal{R}$ be a left-linear constructor TRS over a signature $\mathcal{F} = \mathcal{D} \uplus \mathcal{C}$ and let T^α be a finite set of abstract terms. Let π be a safe argument filtering for T^α in $\mathcal{R}$ that is extended over tuple symbols so that $\pi(f^\sharp) = \pi(f)$ for all $f \in \mathcal{D}$. If there is no infinite $(DP(\mathcal{R}), \mathcal{R}, \pi)$-chain, then $\gamma(T^\alpha)$ is $\leadsto_{\mathcal{R}}$-terminating.*

In order to show the absence of $(DP(\mathcal{R}), \mathcal{R}, \pi)$-chains automatically, we can follow the *DP framework* [20]. In this context, a *DP problem* is a tuple $(\mathcal{P}, \mathcal{R}, \pi)$ where $\mathcal{P}$ and $\mathcal{R}$ are TRSs and π is an argument filtering. If there is no associated infinite $(\mathcal{P}, \mathcal{R}, \pi)$-chain, we say that the DP problem is *finite*. Termination methods are then formulated as *DP processors* that take a DP problem and return a new set of DP problems that should be solved instead.

[5] As in [8], we assume fresh variables in every (occurrence of a) dependency pair and that the domain of substitutions may be infinite.

A DP processor Proc is *sound* if, for all DP problems d, we have that d is finite if all DP problems in $\mathsf{Proc}(d)$ are finite. Therefore, a termination proof starts with the initial DP problem $(DP(\mathcal{R}), \mathcal{R}, \pi)$ and applies sound DP processors until an empty set of DP problems is obtained.

We could adapt most of the standard DP processors in order to deal with the use of data generators and argument filterings following similar ideas as those in [33]. For the sake of brevity, we only present one of such DP processors:

Theorem 4 (argument filtering processor). *Given a DP problem $(\mathcal{P}, \mathcal{R}, \pi)$, let Proc return $\{(\pi(\mathcal{P}), \pi(\mathcal{R}), id)\}$, where $id(\mathsf{f}) = \{1, \ldots, n\}$ for all defined function symbol f/n occurring in $\pi(\mathcal{R})$. Then Proc is sound.*

The nice property of this DP processor is that, after its application, all existing DP processors of [20] for proving the termination of rewriting can also be used for proving the termination of narrowing.

Example 8. Consider the TRS of Example 6, the set of abstract terms $T^\alpha = \{\mathsf{append}(g, v)\}$, and the argument filtering $\pi = \{\mathsf{append} \mapsto \{1\},\ \mathsf{reverse} \mapsto \{1\}\}$ which is safe for T^α. Given this DP problem, the argument filtering processor returns a new DP problem that consists of the following elements:

$$\textit{Dependency pairs:}\quad \begin{cases} \mathsf{APPEND}(\mathsf{cons}(x, xs)) & \to \mathsf{APPEND}(xs) \\ \mathsf{REVERSE}(\mathsf{cons}(x, xs)) & \to \mathsf{REVERSE}(xs) \\ \mathsf{REVERSE}(\mathsf{cons}(x, xs)) & \to \mathsf{APPEND}(\mathsf{reverse}(xs)) \end{cases}$$

$$\textit{Rewrite system:}\quad \begin{cases} \mathsf{append}(\mathsf{nil}) & \to \bot \\ \mathsf{append}(\mathsf{cons}(x, xs)) & \to \mathsf{cons}(x, \mathsf{append}(xs)) \\ \mathsf{reverse}(\mathsf{nil}) & \to \mathsf{nil} \\ \mathsf{reverse}(\mathsf{cons}(x, xs)) & \to \mathsf{append}(\mathsf{reverse}(xs)) \end{cases}$$

Argument filtering: $id = \{\mathsf{append} \mapsto \{1\},\ \mathsf{reverse} \mapsto \{1\}\}$

The derived DP problem can be proved terminating using standard techniques.

4.3 A Transformational Approach

In this section, we present an alternative approach for proving the termination of narrowing. The basic idea is similar to that in the previous section: using an argument filtering to eliminate those subterms that might be bound to a data generator. Now, however, our aim is to transform the original TRS $\mathcal{R}$ into a new TRS $\mathcal{R}'$ so that narrowing terminates in $\mathcal{R}$ if rewriting terminates in $\mathcal{R}'$. As a consequence, *any* termination technique for rewrite systems can be applied to prove the termination of narrowing.

Our transformation is based on the *argument filtering transformation* of [26], that we simplify because, in our case, an argument filtering never returns a single argument position and, moreover, it is only defined over defined function symbols. Roughly speaking, our program transformation generates, for every rule $l \to r$ of the original program,

– a filtered rule $\pi(l) \to \pi_{rhs}(r)$ and
– an additional rule $\pi(l) \to \pi(t)$, for each subterm t of r that is filtered away in $\pi_{rhs}(r)$ and such that $\pi(t)$ is not a constructor term.

Definition 12 (argument filtering transformation). *Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F} = \mathcal{D} \uplus \mathcal{C}$ and let π be an argument filtering over $\mathcal{F}$. The argument filtering transformation AFT_π is defined as follows:*

$$\mathsf{AFT}_\pi(\mathcal{R}) = \pi(\mathcal{R}) \cup \{\pi(l) \to \pi(r') \mid l \to r \in \mathcal{R},\ r' \in \mathsf{dec}_\pi(r),\ \pi(r') \notin \mathcal{T}(\mathcal{C}, \mathcal{V})\}$$

where the auxiliary function dec_π is defined inductively as follows:

$$\begin{array}{rll} \mathsf{dec}_\pi(x) = \varnothing & & (x \in \mathcal{V}) \\ \mathsf{dec}_\pi(c(t_1, \ldots, t_n)) = \bigcup_{i=1}^n \mathsf{dec}_\pi(t_i) & & (c \in \mathcal{C}) \\ \mathsf{dec}_\pi(f(t_1, \ldots, t_n)) = \bigcup_{i \notin \pi(f)} \{t_i\} \ \cup\ \bigcup_{i=1}^n \mathsf{dec}_\pi(t_i) & & (f \in \mathcal{D}) \end{array}$$

Example 9. Consider the TRS $\mathcal{R}$ of Ex. 6. If we consider the argument filtering $\pi_1 = \{\mathsf{append} \mapsto \{1\},\ \mathsf{reverse} \mapsto \{1\}\}$ of Ex. 8, then $\mathsf{AFT}_{\pi_1}(\mathcal{R})$ returns the same filtered rewrite system of Ex. 8.

Consider now the argument filtering $\pi_2 = \{\mathsf{append} \mapsto \{2\},\ \mathsf{reverse} \mapsto \{1\}\}$. Then, $\mathsf{AFT}_{\pi_2}(\mathcal{R})$ returns the following TRS:

$$\begin{array}{rl} \mathsf{append}(y) & \to y \\ \mathsf{append}(y) & \to \mathsf{cons}(\bot, \mathsf{append}(y)) \\ \mathsf{reverse}(\mathsf{nil}) & \to \mathsf{nil} \\ \mathsf{reverse}(\mathsf{cons}(x, xs)) & \to \mathsf{append}(\mathsf{cons}(x, \mathsf{nil})) \\ \mathsf{reverse}(\mathsf{cons}(x, xs)) & \to \mathsf{reverse}(xs) \end{array}$$

Note that the last rule is introduced because we have

$$\mathsf{dec}_{\pi_2}(\mathsf{append}(\mathsf{reverse}(xs), \mathsf{cons}(x, \mathsf{nil}))) = \{\mathsf{reverse}(xs)\}$$

The next result is the main contribution of this section:

Theorem 5. *Let $\mathcal{R}$ be a left-linear constructor TRS and T^α be a finite set of abstract terms, with $T = \gamma(T^\alpha)$. Let π be a safe argument filtering for T^α in $\mathcal{R}$. If $\mathsf{AFT}_\pi(\mathcal{R})$ is terminating, then T is $\rightsquigarrow_{\mathcal{R}}$-terminating.*

The significance of Theorem 5 is that $\mathsf{AFT}_\pi(\mathcal{R})$ can be analyzed using standard techniques and tools for proving the termination of TRSs since no data generator is involved in the derivations of $\mathsf{AFT}_\pi(\mathcal{R})$.

We note that [18] proves that the AFT transformation is subsumed by the DP method regarding simple termination (i.e., termination based on simplification orderings). In our case, the approach of this section is not directly subsumed by that of Sect. 4.2 because we consider termination rather than simple termination. Also, the AFT transformation can be seen as a preprocessing stage so that standard techniques (e.g., the DP method, but not only this method) can be applied to the transformed program, as we will see in the next section.

5 The Termination Tool TNT

In this section, we describe the implementation of a program transformation that follows the approach presented in Sect. 4.3. The tool, called TNT, is publicly available from `http://german.dsic.upv.es/filtering.html`.

The tool is written in Prolog (around 650 lines of code) and includes a parser for TRSs (which accepts the TRS format of the *Termination Problem Data Base*, TPDB, see `http://www.lri.fr/~marche/tpdb/`), a static analysis to infer a safe argument filtering from an abstract term—we consider a single abstract term rather than a set of abstract terms for simplicity—and the AFT_π transformation of Sect. 4.3. The tool is available through a web interface, whose input data are

- a left-linear constructor TRS $\mathcal{R}$ (the user can either write it down or choose it from a selection of TRSs from the TPDB) and
- an initial abstract term t^α that describes a (possibly infinite) set of initial terms $\gamma(t^\alpha)$.

The tool returns a transformed TRS $\mathcal{R}'$ whose termination w.r.t. standard rewriting implies the termination of narrowing for $\gamma(t^\alpha)$ in the original TRS $\mathcal{R}$. The termination of $\mathcal{R}'$ can be analyzed using any tool for proving the termination of rewriting. In particular, the web interface allows the user to check the termination of the transformed TRS using the AProVE tool [19].

For generating a safe argument filtering for a given set of abstract terms, we have adapted a simple *binding-time analysis* [24], which is often used in partial evaluation to propagate static (i.e., ground) and dynamic (i.e., possibly nonground) values through a program. We consider *binding-times* g (ground) and v (possibly variable), rather than the more traditional S (static) and D (dynamic) in the partial evaluation literature (though their meaning is the same). The output of the binding-time analysis is a *division* which includes a mapping $\mathsf{f}/n \mapsto (m_1, \ldots, m_n)$ for every defined function $\mathsf{f}/n \in \mathcal{D}$, where each m_i is a binding-time. A binding-time *environment* is a substitution mapping variables to binding-times. The least upper bound over binding-times is defined as follows:

$$g \sqcup g = g \qquad g \sqcup v = v \qquad v \sqcup g = v \qquad v \sqcup v = v$$

The least upper bound operation can be extended to sequences of binding-times and divisions in the natural way, e.g.,

$$(g, v, g) \sqcup (g, g, v) = (g, v, v)$$
$$\{\mathsf{f} \mapsto (g, v),\ \mathsf{g} \mapsto (g, v)\} \sqcup \{\mathsf{f} \mapsto (g, g),\ \mathsf{g} \mapsto (v, g)\} = \{\mathsf{f} \mapsto (g, v),\ \mathsf{g} \mapsto (v, v)\}$$

Following [24], our binding-time analysis includes two auxiliary functions, B_v and B_e, which are defined in our context as follows:

$$\begin{array}{lll}
B_v[[x]]\ \mathsf{g}/n\ \rho & = (\overbrace{g, \ldots, g}^{n\ times}) & (\text{if } x \in \mathcal{V}) \\
B_v[[\mathsf{c}(t_1, \ldots, t_n)]]\ \mathsf{g}/n\ \rho & = B_v[[t_1]]\ \mathsf{g}/n\ \rho \sqcup \ldots \sqcup B_v[[t_n]]\ \mathsf{g}/n\ \rho & (\text{if } \mathsf{c} \in \mathcal{C}) \\
B_v[[\mathsf{f}(t_1, \ldots, t_n)]]\ \mathsf{g}/n\ \rho & = bt \sqcup (B_e[[t_1]]\ \rho, \ldots, B_e[[t_n]]\ \rho) & (\text{if } \mathsf{f} = \mathsf{g},\ \mathsf{f} \in \mathcal{D}) \\
 & \quad bt & (\text{if } \mathsf{f} \neq \mathsf{g},\ \mathsf{f} \in \mathcal{D}) \\
 & \quad \text{where } bt = B_v[[t_1]]\ \mathsf{g}/n\ \rho \sqcup \ldots \sqcup B_v[[t_n]]\ \mathsf{g}/n\ \rho &
\end{array}$$

$$B_e[[x]]\ \rho = x\rho \qquad \text{(if } x \in \mathcal{V}\text{)}$$
$$B_e[[\mathsf{h}(t_1,\ldots,t_n)]]\ \rho = B_e[[t_1]]\ \rho \sqcup \ldots \sqcup B_e[[t_n]]\ \rho \qquad \text{(if } \mathsf{h} \in \mathcal{C} \cup \mathcal{D}\text{)}$$

Roughly speaking, an expression $(B_v[[t]]\ \mathsf{g}/n\ \rho)$ returns a sequence of n binding-times that denote the (least upper bound of the) binding-times of the arguments of the calls to g/n that occur in t in the context of the binding-time environment ρ. An expression $(B_e[[t]]\ \rho)$ then returns g if t contains no variable which is bound to v in ρ, and v otherwise.

The binding-time analysis is computed as the fixpoint of an iterative process. Assuming that the input abstract term is $\mathsf{f}_1(m_1,\ldots,m_{n_1})$, the initial division is

$$div_0 = \{\mathsf{f}_1 \mapsto (m_1,\ldots,m_{n_1}),\ \mathsf{f}_2 \mapsto (g,\ldots,g),\ \ \ldots,\ \mathsf{f}_k \mapsto (g,\ldots,g)\}$$

where $\mathsf{f}_1/n_1,\ldots,\mathsf{f}_k/n_k$ are the defined functions of the TRS. Then, given a division $div_i = \{\mathsf{f}_1 \mapsto b_1,\ \ \ldots,\ \mathsf{f}_k \mapsto b_k\}$, the next division in the sequence is

$$\begin{aligned} div_{i+1} = \{\ & \mathsf{f}_1 \mapsto b_1 \sqcup B_v[[r_1]]\ \mathsf{f}_1/n_1\ e(b_1,l_1) \sqcup \ldots \sqcup B_v[[r_j]]\ \mathsf{f}_1/n_1\ e(b_j,l_j), \\ & \ldots, \\ & \mathsf{f}_k \mapsto b_k \sqcup B_v[[r_1]]\ \mathsf{f}_k/n_k\ e(b_1,l_1) \sqcup \ldots \sqcup B_v[[r_j]]\ \mathsf{f}_k/n_k\ e(b_j,l_j)\ \} \end{aligned}$$

where $l_1 \to r_1,\ldots,l_j \to r_j$, $j \geq k$, are the rules of $\mathcal{R}$ and the auxiliary function $e(b,l)$ for computing a binding-time environment from a sequence of binding-times and the left-hand side of a rule is defined as follows:

$$e((m_1,\ldots,m_n),\mathsf{f}(t_1,\ldots,t_n)) = \bigcup_{i=1}^{n}\{x \mapsto m_i \mid x \in \mathcal{V}\mathsf{ar}(t_i)\}$$

Once we get a fixpoint, i.e., $div_{i+1} = div_i$ for some $i \geq 0$, the corresponding safe argument filtering π is easily obtained by filtering away the positions of nonground arguments. For instance, if the computed division is

$$div = \{\mathsf{f}_1 \mapsto (m_1^1,\ldots,m_{n_1}^1),\ldots,\mathsf{f}_k \mapsto (m_1^k,\ldots,m_{n_k}^k)\}$$

the corresponding argument filtering is

$$\pi_{div} = \{\mathsf{f}_1 \mapsto \{i \mid m_i^1 = g\},\ldots,\mathsf{f}_k \mapsto \{i \mid m_i^k = g\}\}$$

The fact that π_{div} is a safe argument filtering is a trivial consequence of the fact that the computed division div is *congruent* [24], i.e., of the fact that an argument of a function is classified as g only when every call to this function has a ground term in this argument (according to the computed binding-times).

6 Related Work

Despite the relevance of narrowing as a symbolic computation mechanism, we find in the literature only a few works devoted to analyze its termination.

For instance, Dershowitz and Sivakumar [14] defined a narrowing procedure that incorporates pruning of some unsatisfiable goals. Similar approaches have been presented by Chabin and Réty [10], where narrowing is directed by a graph

of terms, and by Alpuente *et al* [2], where the notion of *loop-check* is introduced to detect some unsatisfiable equations. Also, Antoy and Ariola [4] introduced a sort of memoization technique for functional logic languages so that, in some cases, a finite representation of an infinite narrowing space can be achieved. All these approaches, though, are basically related with *pruning* the narrowing search space rather than analyzing the termination of narrowing.

On the other hand, Christian [11] introduced a characterization of TRSs for which narrowing terminates. Basically, he requires the left-hand sides to be *flat*, i.e., all arguments are either variables or ground terms. Unfortunately, as we discussed at the beginning of Sect. 3, the termination of narrowing for arbitrary terms is quite a strong property that almost no TRS fulfills.

Recent approaches to termination analysis of narrowing include [32, 7]. However, they focused on *quasi-termination* (i.e., whether only finitely many different function calls are reachable) and its application to partial evaluation. Moreover, only needed narrowing and inductively sequential TRSs were considered.

Nishida and Miura [30] adapted the dependency graph method for proving the termination of narrowing. The presented dependency pair method (an extension of that introduced in [31]) is, in principle, not comparable with ours (Sect. 4.2), since we do not allow extra variables in TRSs and they do not remove some (unnecessary) extra-variables of right-hand sides as we do with π_{rhs}.

The closest approach is that of Schneider-Kamp *et al* [33], who presented an automated termination analysis for logic programs. In their approach, logic programs are first translated into TRSs and, then, logic variables are replaced by possibly infinite terms. An extension of the dependency pair framework for dealing with argument filterings is presented, which is similar to our extension in Sect. 4.2. Besides considering a different target (proving termination of SLD resolution *vs* proving termination of narrowing), there are a number of differences between both approaches. First, [33] considers the replacement of logic variables by infinite terms, while we use data generators (so that we could reuse existing results relating narrowing and standard *finitary* rewriting). Also, they consider arbitrary argument filterings but require the *variable condition* (i.e., that the filtered TRS contains no extra variables). In our case, argument filterings must be safe which, in principle, do not always imply that the variable condition holds in filtered TRSs. Actually, we allow extra variables above the defined functions of the right-hand sides of the filtered rules (which are then replaced by $\bot$ in π_{rhs} since they play no role for termination in our context). Furthermore, we introduce a simple binding-time analysis in order to automate the generation of safe argument filterings from higher-level abstract terms. Finally, we also present a transformational approach to proving termination, while [33] focuses on a direct approach based on the dependency pair framework.

7 Conclusions

We have presented in this paper new techniques for proving the termination of narrowing in left-linear constructor systems. Our approach allows one to an-

alyze the termination of narrowing by analyzing the termination of rewriting, so that one can reuse existing methods and tools in the extensive literature on termination of rewriting.

Regarding future work, we find it interesting to investigate the application of our results in order to improve the precision of narrowing-driven partial evaluation [32]. Also, it would be useful to extend our approach in order to accept source Curry programs rather than TRSs.

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